Announcements

1) #3 HW 2

$[0,1]$ should be $[a,b]$

2) #7 b is now 5 points instead of 10

3) Good set of notes for last class online

Theorem: (Lebesgue)

Let $f: [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable on $[a,b]$ if and only if D has measure zero.

Proof: We showed $\Leftarrow$

$\Rightarrow$ Let $\alpha > 0$, $\varepsilon > 0$.

We assume f is Riemann integrable on $[a,b]$. Then $\exists$ a partition P of $[a,b]$ with

$$U(f,P) - L(f,P) < \varepsilon \cdot \alpha$$

Let $P = \{z_1, z_2, \dots, z_n\}$ with $a = z_1 < z_2 < \cdots < z_n = b$.

If $x \in D_\alpha \cap (z_i, z_{i+1})$

for some i, $1 \leq i < n$,

then $\exists$ $y, z \in (z_i, z_{i+1})$

with $|f(y) - f(z)| \geq \alpha$.

In this case,

$$M_i - m_i \geq |f(y) - f(z)| \geq \alpha.$$

Then let

$$S = \{ i \mid D_\alpha \cap (z_i, z_{i+1}) \neq \{\phi\} \}.$$

We have

$$\varepsilon\alpha > U(f,P) - L(f,P)$$

$$= \sum_{i=1}^{n-1} (M_i - m_i)(z_{i+1} - z_i)$$

$$\geq \sum_{i\in S} (M_i - m_i)(z_{i+1} - z_i)$$

$$\geq \alpha \sum_{i\in S} (z_{i+1} - z_i)$$

$$\Rightarrow \varepsilon > \sum_{i\in S} (z_{i+1} - z_i).$$

Therefore,

$$D_\alpha \subseteq \underbrace{\bigcup_{i\in S} (z_i, z_{i+1})}_{\text{sum of lengths is less than } \varepsilon} \cup \underbrace{\{z_i \mid 1 \le i \le n\}}_{\text{finite, so measure zero}}$$

Since ε is arbitrary, we conclude D_α has measure zero.

This implies

$D_{\frac{1}{n}}$ has measure zero

for all $n \in \mathbb{N}$. But

$D = \bigcup_{n=1}^{\infty} D_{\frac{1}{n}}$ and

the countable union of

measure zero sets is

measure zero, so D

has measure zero.

$\square$

Example 1: (Dirac measure).

Choose $x \in \mathbb{R}$. Define a set $S \subseteq \mathbb{R}$ to have Dirac x-measure zero if $x \notin S$.

In this case, $(-\infty, x) \cup (x, \infty)$ has measure zero!

The Fundamental Theorem of Calculus

You are familiar with this theorem for continuous integrands.

We will establish the theorem for f bounded and integrable on $[a, b]$.

Fundamental Theorem

Let f be bounded and integrable on $[a,b]$.

1) If g is any function $g: \mathbb{R} \to \mathbb{R}$ with $g'(x) = f(x)$ $\forall\, x \in [a,b]$, then

$$\int_a^b f(x)\,dx = g(b) - g(a).$$

2) If we set

$$h(x) = \int_a^x f(t)\,dt,$$ then

h is uniformly continuous on $[a,b]$. If f is continuous at $x \in [a,b]$, then h is differentiable at x and $h'(x) = f(x)$.

proof: 1) Let

$P = \{ x_i \mid 1 \le i \le n \}$ be

a partition of $[a,b]$,

$a = x_1 < x_2 < \cdots < x_n = b$.

Since g is differentiable on

$[a,b]$, it is differentiable

on $[x_i, x_{i+1}]$ $\forall\ 1 \le i < n$.

We then use the Mean Value Theorem on each interval $[x_i, x_{i+1}]$: $\exists\, c_i \in [x_i, x_{i+1}]$,

$$\frac{g(x_{i+1}) - g(x_i)}{x_{i+1} - x_i} = g'(c_i)$$

$$= f(c_i).$$

Hence

$$L(f,P) = \sum_{i=1}^{n-1} m_i (x_{i+1} - x_i)$$

$$\leq \sum_{i=1}^{n-1} f(c_i)(x_{i+1} - x_i)$$

$$\leq \sum_{i=1}^{n-1} M_i (x_{i+1} - x_i)$$

$$= U(f,P).$$

But $f(c_i) = \dfrac{g(x_{i+1}) - g(x_i)}{x_{i+1} - x_i}$,

so

$$\sum_{i=1}^{n-1} f(c_i)(x_{i+1} - x_i)$$

$$= \sum_{i=1}^{n-1} \frac{g(x_{i+1}) - g(x_i)}{\cancel{x_{i+1} - x_i}} \cdot \cancel{(x_{i+1} - x_i)}$$

$$= g(b) - g(a).$$

Then for any partition P,

$$U(f,P) \geq g(b)-g(a) \geq L(f,P)$$

But since f is integrable,

$\forall\ \varepsilon>0$, we can choose

P with $U(f,P)-L(f,P)<\varepsilon$

$$\Rightarrow \int_a^b f(x)\,dx = g(b)-g(a)$$

2) $h(x) = \int_a^x f(t)\,dt.$

Since f is bounded, $\exists\ M > 0$,

$|f(t)| < M \quad \forall\ t \in [a,b]$

(suppose $x > y$)

Then $|h(x) - h(y)|$

$$= \left| \int_a^x f(t)\,dt - \int_a^y f(t)\,dt \right|$$

$$= \left| \int_y^x f(t)\,dt \right|$$

But

$$\left| \int_y^x f(t)\,dt \right|$$

$$\le \int_y^x |f(t)|\,dt$$

$$< \int_y^x M\,dt$$

$$= M\,(x-y).$$ If $y>x$,

we would have arrived at

$$|h(x)-h(y)| < M(y-x).$$

Then

$$|h(x) - h(y)| < M|x-y|$$

$\forall\ x, y \in [a,b]$, so

h is Lipschitz and

hence uniformly continuous

on $[a,b]$.

Last part Monday